MIT OCW 8.04 QM I - Solutions to Problems Sets

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Problem Set 1

Solution 1:

(a) The period of an orbit is given by

$$\frac{e^2}{r^2} = m\omega^2 r \implies T = \frac{2\pi}{e}\sqrt{mr^3}$$

so in one period, the change in energy is:

$$\Delta E = \frac{-2}{3}\frac{e^2 a^2}{c^3}\left(\frac{2\pi}{e}\sqrt{m_e r^3}\right) = \frac{-4\pi}{3}\frac{e^5}{m_e^2 r^4 c^3}\sqrt{m_e r^3}$$

The kinetic energy is given by:

$$K = \frac{e^2}{2r} = 3.552 \cdot 10^{-6}$$

so:

$$\frac{\Delta E}{K} = \frac{-8\pi}{3}\frac{e^3}{m_e^2 r^3 c^3}\sqrt{m_e r^3}$$

item The speed is given by:

$$v = e\sqrt{\frac{1}{mr}}$$

- 50 pm: 2.25×10^6 m/s
- 1 pm: 1.59×10^7 m/s
- 1 fm: 5.04×10^7 m/s

(b) Ignoring relativistic corrections is justified, since even with the speed of the electron a tenth of that of light, the Lorentz factor γ is only around 1.005. By noting that $E = \frac{1}{2}V = -\frac{e^2}{2r}$ using the Virial theorem, and that $a = \frac{F_e}{m_e} = \frac{e^2}{m_e r^2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{e^2}{2r}\right) = -\frac{2}{3}\frac{e^2}{c^3}\left(\frac{e^2}{m_e r^2}\right)^2$$

(c) According to this model, there is no minimum energy that the electron can have. As r approaches zero, the electron's energy will approach $-\infty$.

Solution 2:

(a) In any system where the force obeys an inverse square law, there will be a relationship between kinetic and potential energy called the viral theorem. We know from the Virial theorem that the kinetic energy

$$\langle K \rangle = -\frac{1}{2}\langle V \rangle > 0.$$

Hence, the total energy is half the Coulomb potential

$$E = \langle K \rangle + \langle V \rangle = -\langle K \rangle = \frac{1}{2}\langle V \rangle < 0$$

up to sign conventions, this is synonymous with the Rydberg energy.

(b) We know that

$$K = \frac{L^2}{2I} = \frac{L^2}{2m_e r_n^2}.$$

However,

$$K = -\frac{1}{2}V = \frac{1}{2}\frac{e^2}{r}.$$

Substituting the angular momentum quantization condition $L_n = n\hbar$, $\frac{n^2\hbar^2}{2m_e r_n^2} = \frac{1}{2}\frac{e^2}{r_n}$.

$$r_n = \frac{n^2\hbar^2}{m_e e^2} = \frac{n^2\hbar^2}{(\hbar c)m_e(\frac{e^2}{\hbar c})} = n^2 \cdot \frac{\hbar}{m_e c} \cdot \frac{e^2}{\hbar c} = \frac{\lambda_c}{\alpha}n^2$$

$$E_n = \frac{1}{2}V = \frac{e^2}{2r_n} = \frac{\alpha e^2}{2n^2\lambda_c} = \frac{\alpha^2 E_e}{2n^2}$$

Solution 3:

(a) (i) $\lambda = \frac{h}{p} = \frac{h}{mv} \approx 1.5 \times 10^{-38}$ m

(ii) $\lambda = \frac{h}{p} = \frac{h}{mv} \approx 6.6 \times 10^{-31}$ m

(iii) $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \approx 1.9 \times 10^{-16}$ m

(iv) $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \approx 2.7 \times 10^{-8}$ m

(b) (i) $\nu = c/\lambda$. Thus, the range of frequencies is from 4.3×10^{14} Hz to 7.5×10^{14} Hz

(ii) Let P be the power of operation and ν the frequency of operation. Then, the energy of each photon is $E = h\nu$, so the numbers of photons emitted per second is $\frac{P}{h\nu}$. Note that $\nu = \frac{c}{\lambda}$. Plugging in numbers, a microwave emits 1.8×10^{26} photons per second, a low-power laser around 3.2×10^{16} photons per second, and a cell phone around 4.5×10^{23} photons per second.

(iii) The energy required to heat the water is $\Delta E = mc\Delta T = \rho V c\Delta T \approx 8000$ J, where c is the specific heat capacity, ρ is the density, and V is the volume. Each photon has an energy of $E_\gamma = h\nu \approx 1.7 \times 10^{-24}$ J. Thus, the number of photons required to heat up the water by the given temperature difference is $\frac{\Delta E}{E_\gamma} \approx 4.8 \times 10^{27}$ microwave photons (based on frequency from previous subpart).

(iv) A classical description would work better for radio waves than for X-rays. Since radio wave photons would have lower energies than X-ray photons, relativistic effects would cause less deviation from the classical wave description.

Solution 4:

(a) We use the Taylor expansion of $e^{i\theta}$.

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \dots \\ &= 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 + \dots \end{aligned}$$

Separating these functions into even and odd powers gives us

$$\begin{aligned}
e^{i\theta} &= \left(1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 + \dots\right) + i\left(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots\right) \\
e^{i\theta} &= \cos\theta + i\sin\theta.
\end{aligned}$$

(b) a is the real part of the complex exponential, while b is the imaginary part. Using Euler's formula:

$$\begin{aligned}
a &= r\cos\theta \\
b &= r\sin\theta
\end{aligned}$$

Then, note that the magnitude of the complex exponential is just the value of r. Additionally, the angle θ can be considered using the ratio of imaginary to real components of the complex exponential.

$$\begin{aligned}
r &= \sqrt{a^2 + b^2} \\
\theta &= \arctan\frac{b}{a}
\end{aligned}$$

(c) **(i)** Note that $i = e^{\frac{i\pi}{2}}$, since $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$. Thus, using the properties of exponents:

$$\begin{aligned}
re^{i\theta} \cdot i &= e^{i\theta} \cdot e^{\frac{i\pi}{2}} \\
&= re^{i(\theta + \frac{\pi}{2})}
\end{aligned}$$

(ii) $iz = i(a + ib) = ia - b$. Thus, the real part of iz is $-b$.

(d) **(i)** 0, since its complex conjugate and the negative of its complex conjugate are both equal to itself, 0.

(ii) The complex conjugate flips the sign of the imaginary term, so taking the complex conjugate twice leaves us with $(z^*)^* = z$. For the second part of the question, note that $re^{-i\theta} = r\cos -\theta + i\sin -\theta$. Since cosine is an odd function and sine is an odd function, $re^{-i\theta} = r\cos\theta - i\sin\theta = a - bi$, which is the conjugate of the original complex number.

(iii) The real part is the same and the imaginary part becomes negative, by the definition of the complex conjugate.

(iv) $zz^* = (a + bi)(a - bi) = a^2 + b^2$. Using complex exponentials, $zz^* = re^{i\theta} * re^{-i\theta} = r^2$. Since $r = |z|$, $zz^* = |z|^2$.

(e) First, consider the double angle identities. We can represent $\cos 2\theta$ and $\sin 2\theta$ by the real and imaginary parts of $e^{2i\theta} = e^{i\theta} * e^{i\theta}$. Expanding:

$$\begin{aligned}
e^{2i\theta} &= e^{i\theta} \cdot e^{i\theta} \\
&= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) \\
&= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta \\
&= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta).
\end{aligned}$$

Taking the real and imaginary parts of this expression, we obtain the double angle identities:

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta
\end{aligned}$$

The same thing can be done to solve for the triple angle identities, by finding the real and imaginary components of $e^{3i\theta} = e^{i\theta} * e^{i\theta} * e^{i\theta}$

$$\begin{aligned}
e^{3i\theta} &= e^{i\theta} \cdot e^{i\theta} \cdot e^{i\theta} \\
&= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta) \\
&= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\
&= (4\cos^3\theta - 3\cos\theta) + i(3\sin\theta - 4\sin^3\theta)
\end{aligned}$$

Taking the real and imaginary components, we obtain the triple angle identities:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

Now, the more general case of such identities can be found, with two arbitrary angles A and B added with each other in a similar fashion.

$$\begin{aligned}
e^{i(A+B)} &= e^{iA}\cdot e^{iB} \\
&= (\cos A + i\sin A)(\cos B + i\sin B) \\
&= \cos A\cos B + i\sin A\cos B + i\sin B\cos A - \sin A\sin B \\
&= (\cos A\cos B - \sin A\sin B) + i(\sin A\cos B + \sin B\cos A)
\end{aligned}$$

Taking the real and imaginary components again, we obtain the sum and difference identities:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \sin B\cos A$$

Solution 5: Conservation of momentum and energy gives us (neglecting powers of c):

$$p_\gamma + p_1 = p_2$$
$$p_\gamma + \sqrt{m_e^2 + p_1^2} = \sqrt{m_e^2 + p_2^2}$$

Squaring and solving will give us:

$$(p_2 - p_1)^2 - (p_2^2 - p_1^2) + 2(p_2 - p_1)\sqrt{(p_1^2 + m_e^2)} = 0$$

One obvious solution is $p_2 - p_1 = 0$ but this is only achieved when $p_\gamma = 0$ which cannot be possible. Dividing through by this factor, we are left with:

$$\sqrt{(p_1^2 + m_e^2)} - p_1 = 0$$

which can only be possible if $m_e = 0$ which it certainly is not.

Solution 6:

(a) Adding a phase shift to the bottom leg is equivalent to effectively multiplying the state by the matrix:

$$\text{PS} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}$$

The output is represented by:

$$\begin{aligned}
\text{output} &= (\text{BS2})(\text{PS})(\text{BS1})\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\delta} \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & e^{i\delta} \\ 1 & -e^{i\delta} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} -1 + e^{i\delta} & 1 + e^{i\delta} \\ -1 - e^{i\delta} & 1 - e^{i\delta} \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} (-1 + e^{i\delta})\alpha + (1 + e^{i\delta})\beta \\ -(1 + e^{i\delta})\alpha + (1 - e^{i\delta})\beta \end{pmatrix}
\end{aligned}$$

The squares of the norms give the respective probabilities:

$$P_0 = \frac{1 + (2\beta^2 - 1)\cos\delta}{2}$$
$$P_1 = \frac{1 - (2\beta^2 - 1)\cos\delta}{2}$$

(b) Due to symmetry, this is equivalent to simply switching β and α, or effectively flipping the setup so that P_0 and P_1 swap and β and α swap. Therefore, the new probabilities are:

$$P_0 = \frac{1 - (1 - 2\beta^2)\cos\delta}{2}$$
$$P_1 = \frac{1 + (1 - 2\beta^2)\cos\delta}{2}$$

(c) By placing a phase shift at the top and bottom, it is equivalent to multiplying by

$$\text{PS} = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix} = e^{i\delta}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore, all this does is shift the final phase of both by rotating the state by an angle δ, which will not change the overall probability. This will lead to the result derived in the lecture:

$$P_0 = \beta^2$$
$$P_1 = \alpha^2$$

Solution 7:

(a) Consider several rounds of testing. Suppose we start off with N working bombs to test. After the first round, there will be $N/2$ explosions, $N/4$ verified bombs, and $N/4$ bombs that still need to be verified. Recursively, we can represent the number of bombs that get certified each round as:

$$f_{N+1} = \frac{1}{4} f_N$$

The total sum of certified bombs would be:

$$\frac{N}{4}\frac{1}{1 - \frac{1}{4}} = \frac{N}{3}$$

So the total fraction of bombs that get certified is:

$$\frac{1}{3}$$

(b) The Bayesian formula tells us the probability of it being defective is:

$$P(\text{defective}|D_0) = \frac{P(D_0|\text{defective})P(\text{defective})}{P(D_0)}$$
$$= \frac{(1)(0.8)}{(0.8)(1) + (0.2)(0.25)}$$
$$= 94.1\%$$

Problem Set 2

Solution 1:

(a) In SI units, the de Broglie wavelength of a nonrelativstic electron is $\lambda_{nr} = \frac{h}{p} = \frac{h}{\sqrt{2mE_{kin}}}$. To make computation easier, let us consider an arbitrary energy value, say 1×10^{-12} J. Using the SI formula, the de Broglie wavelength associated with an electron of this energy is roughly $\lambda \approx 4.9 \times 10^{-19}$ m. 1×10^{-12} J is about $6.2MeV$. Noting the conversion from the angstrom to the meter, $\delta \approx 12.3$.

Alternatively, we can show the relationship symbolically. Using the fact that $\lambda p = h$, we can write δ as:

$$\delta = \lambda\sqrt{kE} \times 10^{10} = \lambda p\sqrt{\frac{k}{2m_e}} \times 10^{10} = 12.3$$

where k is the conversion factor from mass to energy.

(b) The de Broglie wavelength is:

$$\lambda = \frac{h}{p}$$

where the relativistic momentum p is given by

$$p^2c^2 = E^2 - m^2c^4 \implies p = mc\sqrt{\gamma^2 - 1}$$

Combined together, we get the same form which is asked in the question:

$$\lambda_{\mathrm{r}} = \frac{h}{mc}\frac{1}{\sqrt{\gamma^2 - 1}}$$

where $\ell = \frac{h}{mc} = 2430$ fm

(c) For a non-relativistic electron, the de Broglie wavelength is

$$\lambda_{\mathrm{nr}} = \frac{h}{mc(v/c)} = \frac{\ell}{\beta}$$

where β is given by:

$$\gamma^2 = \frac{1}{1-\beta^2} \implies \frac{1}{\beta} = \frac{\gamma}{\sqrt{\gamma^2 - 1}}$$

giving

$$\lambda_{\mathrm{r}} = \frac{h}{\gamma m v} = \frac{\ell}{\sqrt{\gamma^2 - 1}}$$

Interestingly enough, $\ell = \frac{h}{mc}$ is just the Compton wavelength! Another interesting, though not at all surprising conclusion, is that the de Broglie wavelength transforms under the Lorentz transformations in a similar way length does (since mv turns into γmv).

(d) (i) This can only be possible if $\lambda_r = \ell$. Using the relativistic formula, we must have:

$$\gamma^2 - 1 = 1 \implies \beta \approx 0.71$$

which is relativistic.

(ii) First, note that $p = \sqrt{(E/c)^2 + m^2c^2}$. Using this relation, the momenta of the electron and proton can be calculated, allowing us to find the de Broglie wavelength.

$$\lambda_e = \frac{h}{p} = \frac{h}{\sqrt{(E/c)^2 + m_e^2c^2}} \approx 1.24 \times 10^{-18} \text{ m}$$

$$\lambda_p = \frac{h}{p} = \frac{h}{\sqrt{(E/c)^2 + m_p^2c^2}} \approx 1.77 \times 10^{-19} \text{ m}$$

(iii) Based on our results from part c), $\gamma = 1.1$. This gives

$$\beta = \frac{\sqrt{21}}{11} \approx 0.42$$

Computing, the kinetic energy,

$$T = \gamma E_{\text{rest}} \approx 0.562 \text{ MeV}$$

.

Solution 2:

(a) Noting the units of the given constants and doing some dimensional analysis yields

$$a_0 \sim \frac{\hbar^2}{m_e e^2} \approx 5.3 \times 10^6 \text{ fm}$$

The easiest way to observe this result is to note we can create two quantities of energy:

$$E_1 = \frac{\hbar^2}{m_e a_0^2}$$
$$E_2 = \frac{e^2}{a_0}$$

Taking their ratio directly gives a_0. We know this is the unique theorem by applying the Buckingham Pi theorem. Note that in Gaussian units (cgs), we only have three dimensions instead of the usual four. With four distinct quantities, we can only make one unique meaningful dimensionless factor.

(b)

$$a_0 \alpha = \frac{\hbar^2}{m_e e^2} \frac{e^2}{\hbar c} = \frac{\hbar}{m_e c} = \lambda_c$$
$$\lambda_c \alpha = \frac{\hbar}{m_e c} \frac{e^2}{\hbar c} = \frac{e^2}{m_e c^2} = r_0$$

Thus,

$$\lambda_c = \alpha a_0 \approx 3.9 \times 10^4 \text{ fm}$$
$$r_0 = \alpha \lambda_c \approx 2.8 \times 10^2 \text{ fm}$$

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Solution 3: We can solve this generally, but before we do so, we need to state and prove the following lemmas:

> **Lemma 1**: If $A = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a 1×2 matrix then $A^\dagger A = |u_1|^2 + |u_2|^2$

> **Proof**: By definition, the Hermitian $A^\dagger$ can be written as $\begin{pmatrix} u_1^* & u_2^* \end{pmatrix}$ so their product gives:
>
> $$\begin{pmatrix} u_1^* & u_2^* \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1^* u_1 + u_2^* u_2 = |u_1|^2 + |u_2|^2$$

Thus, when the question acts for the conservation of probability, an equivalent statement is to set the equality

$$u^\dagger u = u'^\dagger u' = 1$$

> **Lemma 2**: If R is a 2×2 matrix and u is a 1×2 matrix, then $(Ru)^\dagger = u^\dagger R^\dagger$

Proof: Generalize our matrices as $R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Then:

$$\begin{aligned}(Ru)^\dagger &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right]^\dagger \\ &= \begin{pmatrix} au_1 + bu_2 \\ cu_1 + du_2 \end{pmatrix}^\dagger \\ &= \begin{pmatrix} a^*u_1^* + b^*u_2^* & c^*u_1^* + d^*u_2^* \end{pmatrix}\end{aligned}$$

and similarly

$$\begin{aligned}u^\dagger R^\dagger &= \begin{pmatrix} u_1^* & u_2^* \end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \\ &= \begin{pmatrix} a^*u_1^* + b^*u_2^* & c^*u_1^* + d^*u_2^* \end{pmatrix}\end{aligned}$$

With these out of the way, the proof becomes trivial. All we need to show is that:

$$\begin{aligned}u'^\dagger u' &= (Ru)^\dagger(Ru) \\ &= u^\dagger(R^\dagger R)u\end{aligned}$$

If we wish to end up with the equality:

$$u^\dagger u = u'^\dagger u' = 1$$

then it must be true that $R^\dagger R = I$, or in other words: R is unitary.

Solution 4:

(a) The matrix of the beam splitter can be represented like so: $\text{BS} = \begin{pmatrix} r & t \\ t & -r \end{pmatrix}$ The matrix can be shown to be unitary by calculating its conjugate transpose (which in this case, happens to be itself) and multiplying the two matrices, which ultimately results in the identity matrix

(b) **Defective bomb:** We can simply apply the BS matrix to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (which represents the photon coming from the top). Thus:

$$\begin{pmatrix} P_0 \\ P_1 \end{pmatrix} = \begin{pmatrix} r & t \\ t & -r \end{pmatrix}\begin{pmatrix} r & t \\ t & -r \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Working bomb: First let's see what happens after the light goes through the first beam splitter. We apply the BS matrix to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus:

$$\begin{pmatrix} r & t \\ t & -r \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} r \\ t \end{pmatrix}$$

We will now consider two cases:

- Case 1: Bomb does not Explode - This has a chance of r^2 of happening and can be thought of as sending a $\begin{pmatrix} r \\ 0 \end{pmatrix}$ beam through the second BS matrix. This leads to athe state of the exit beam being

$$\begin{pmatrix} r^2 \\ rt \end{pmatrix} = \begin{pmatrix} r^2 \\ r\sqrt{1-r^2} \end{pmatrix}$$

- Case 2: Bomb explodes - rip gg

Therefore, the probabilities are thus

$$\begin{pmatrix} P_0 \\ P_1 \end{pmatrix} = \begin{pmatrix} r^4 \\ r^2 - r^4 \end{pmatrix} = \begin{pmatrix} R^2 \\ R - R^2 \end{pmatrix}$$

(c) We can only reasonably certify a working bomb if it ends up striking the detector D_1. This has a probability of $p_1 = R - R^2$ of happening. Some bombs may explode and for the rest, there is a probability of $p_0 = R^2$ of striking D_0. If we start with N bombs, then after the first round we will need to test the remaining $p_0 N$ bombs, of which $p_0 p_1 N$ will be certified. This gives the recursive series:

$$p_1(1 + p_0 + p_0^2 + p_0^3 \cdots) = \frac{p_1}{1 - p_0} = \frac{R - R^2}{1 - R^2} = \frac{R}{R + 1}$$

This reaches a maximum as $R \to 1$ but $R \neq 1$. We can interpret this physically as: although we can verify almost half of all the bombs, it may take an extremely long time.

Solution 5: A Taylor expansion gives us, letting $\phi \equiv kx - \omega t$

$$\begin{aligned}\cos(\phi + \epsilon) + \gamma \sin(\phi + \epsilon) &= \cos\phi - \epsilon \sin\phi + \gamma \sin\phi + \epsilon\gamma \cos\phi \\ &= (1 + \gamma\epsilon)\cos\phi + (\gamma - \epsilon)\sin\phi\end{aligned}$$

Setting this equal to

$$a\cos\phi + a\gamma \sin\phi$$

allows us to compare coefficients, giving us two equations:

$$\begin{aligned}1 + \gamma\epsilon &= a \\ \gamma - \epsilon &= a\gamma\end{aligned}$$

Solving this gives:

$$\gamma = \pm i$$

and

$$a = 1 \pm \gamma\epsilon$$

We understand that the conventional description for a plane matter wave is:

$$\Psi_0 = e^{i\phi}$$

such that if the phase is shifted by ϵ, then

$$\Psi = e^{i\phi} e^{i\epsilon} = \Psi_0(1 + i\epsilon)$$

Therefore, the only solution that describes the behaviour is $\gamma = i$.

Problem Set 3

Solution 1: For all of these parts, we can assume the operators to act on some arbitrary function $\phi(x)$

(a)

$$\begin{aligned}[A, BC] &= [A, B]C + B[A, C] \\ [A, BC]\phi(x) &= [A, B]C\phi(x) + B[A, C]\phi(x) \\ ABC\phi(x) - BCA\phi(x) &= ABC\phi(x) - BAC\phi(x) + BAC\phi(x) - BCA\phi(x)\end{aligned}$$

After simplifying, it is clear that both sides of this identity are the same, and that it is in fact true.

(b)

$$\begin{aligned}[AB, C] &= A[B, C] + [A, C]B \\ [AB, C]\phi(x) &= A[B, C]\phi(x) + [A, C]B\phi(x) \\ ABC\phi(x) - CAB\phi(x) &= ABC\phi(x) - ACB\phi(x) + ACB\phi(x) - CAB\phi(x)\end{aligned}$$

After simplifying, it is clear that both sides of this identity are the same, and that it is in fact true.

(c) Expanding $[A, [B, C]] + [B, [C, A]] + [C, [A, B]]$, we get:

$$\begin{aligned}&A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\ =\ &ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC\end{aligned}$$

All the terms cancel out and the final result for the expression is zero.

(d) First, note that:

$$\hat{x}^n\phi(x) = x^n\phi(x), \hat{p}^n\phi(x) = (-i\hbar)^n\frac{\partial^n\phi(x)}{\partial x^n}$$

Thus,

$$\begin{aligned}[\hat{x}^n, \hat{p}]\phi(x) &= -x^n i\hbar\frac{\partial\phi(x)}{\partial x} + i\hbar\frac{\partial}{\partial x}(x^n\phi(x)) \\ &= -x^n i\hbar\frac{\partial\phi(x)}{\partial x} + i\hbar(nx^{n-1}\phi(x) + x^n\frac{\partial\phi(x)}{\partial x}) \\ &= i\hbar nx^{n-1}\phi(x) \\ &\implies [\hat{x}^n, \hat{p}] = i\hbar nx^{n-1}\end{aligned}$$

For the next consideration:

$$\begin{aligned}[\hat{x}, \hat{p}^n]\phi(x) &= x(-h)^{\bar{}n}\frac{\partial^n\phi}{\partial x^n} - (i\hbar)^n\frac{\partial^n}{\partial x^n}(x\phi) \\ &= (-i\hbar)^n\left[x\frac{\partial^n\phi}{\partial x^n} - \frac{\partial^n}{\partial x^n}(x\phi)\right]\end{aligned}$$

We need to evaluate: $\frac{\partial^n}{\partial x^n}(x\phi)$. This is easiest seen by starting with $n = 1$, giving:

$$\frac{\partial}{\partial x}(x\phi) = \phi + x\frac{\partial\phi}{\partial x}$$

Letting $n = 2$, we get:

$$\frac{\partial^2}{\partial x^2}(x\phi) = 2\frac{\partial\phi}{\partial x} + x\frac{\partial^2\phi}{\partial x^2}$$

This suggests the pattern:

$$\frac{\partial^n}{\partial x^n}(x\phi) = n\frac{\partial^{n-1}\phi}{\partial x^{n-1}} + x\frac{\partial^n\phi}{\partial x^n}$$

We can prove this via induction. Suppose this is true for $n = n$. Then for $n = n + 1$:

$$\begin{aligned}\frac{\partial^{n+1}}{\partial x^{n+1}}(x\phi) &= \frac{\partial}{\partial x}\left[n\frac{\partial^{n-1}\phi}{\partial x^{n-1}} + x\frac{\partial^n \phi}{\partial x^n}\right] \\ &= n\frac{\partial^n\phi}{\partial x^n} + \frac{\partial^n\phi}{\partial x^n} + x\frac{\partial^{n+1}\phi}{\partial x^{n+1}} \\ &= (n+1)\frac{\partial^n\phi}{\partial x^n} + x\frac{\partial^{n+1}\phi}{\partial x^{n+1}}\end{aligned}$$

proving our statement. Plugging this expression into our general equation allows us to simplify it to:

$$[\hat{x}, \hat{p}^n] = -n(-i\hbar)^n \frac{\partial^{n-1}}{\partial x^{n-1}} = i\hbar n \hat{p}^{n-1}$$

(e)

$$\begin{aligned}[\hat{x}\hat{p}, \hat{x}^2]\phi(x) &= \hat{x}\hat{p}\hat{x}^2\phi(x) - \hat{x}^2\hat{x}\hat{p}\phi(x) \\ &= x\left(-i\hbar\frac{\partial}{\partial x}\left(x^2\phi(x)\right)\right) - x^3\left(-i\hbar\frac{\partial\phi(x)}{\partial x}\right) \\ &= -2i\hbar x^3\phi(x) - i\hbar x^3\frac{\partial\phi(x)}{\partial x} + i\hbar x^3\frac{\partial\phi(x)}{\partial x} \\ &= -2i\hbar x^3\phi(x) \\ &\implies [\hat{x}\hat{p}, \hat{x}^2] = -2i\hbar x^3\end{aligned}$$

$$\begin{aligned}[\hat{x}\hat{p}, \hat{p}^2]\phi(x) &= \hat{x}\hat{p}\hat{p}^2\phi(x) - \hat{p}^2\hat{x}\hat{p}\phi(x) \\ &= x(-i\hbar)^3\frac{\partial^3\phi(x)}{\partial x^3} - (-i\hbar)^3\frac{\partial^2}{\partial x^2}\left(x\frac{\partial\phi(x)}{\partial x}\right) \\ &= -i\hbar^3 x\frac{\partial^3\phi(x)}{\partial x^3} + i\hbar^3 x\frac{\partial^3\phi(x)}{\partial x^3} + 2i\hbar^3\frac{\partial^2\phi(x)}{\partial x^2} \\ &= 2i\hbar^3\frac{\partial^2\phi(x)}{\partial x^2}\end{aligned}$$

Noting the definition of the momentum operator, we can conclude:

$$[\hat{x}\hat{p}, \hat{p}^2] = -2i\hbar\hat{p}^2$$

Solution 2: The main argument of the stationary phase principle is that if the phase changes too quickly, the integral will cancel out over the function. Thus, where the function peaks, $\frac{\partial\phi}{\partial k} = 0$, where ϕ is the phase.

(a)

$$\Psi(x) = \int_{-\infty}^{\infty} \mathrm{d}k\ \exp\left(-L^2(k-k_0)^2 + ikx\right) = \int_{-\infty}^{\infty} \mathrm{d}k\ \exp\left(-L^2k^2 + 2L^2kk_0 - L^2k_0^2 + ikx\right)$$

The phase part of the exponent is the imaginary term, so:

$$\frac{\partial}{\partial k}(ikx) = ix = 0 \implies x = 0$$

This can be confirmed by actually evaluating the integral:

$$\begin{aligned}\Psi(x) &= \int_{-\infty}^{\infty} \mathrm{d}k\ \exp\left(-L^2k^2 + (2L^2k_0 + ix)k - L^2k_0^2\right) \\ &= \sqrt{\frac{\pi}{L^2}}\exp\left(L^2k_0^2\right)\exp\left((2L^2k_0 + ix)^2/4L^2\right) \\ &= \frac{\sqrt{\pi}}{L}\exp\left(L^2k_0^2\right)\exp\left(L^2k_0^2\right)\exp(ik_0x)\exp\left(-x^2/4L^2\right)\end{aligned}$$

Notice that the last exponential term decays to zero as x moves away from zero, proving our conclusion that the wavefunction peaks at $x = 0$

(b)

$$\Psi(x) = \int_{-\infty}^{\infty} \mathrm{d}k \ \exp\big(-L^2(k-k_0)^2 - ikx_0 + ikx\big)$$

The wave-function peaks when

$$\frac{\partial}{\partial k}(-ikx_0 + ikx) = -ix_0 + ix \implies x = x_0$$

Let us try to verify this by evaluating the integral:

$$\begin{aligned}
\Psi(x) &= \int_{-\infty}^{\infty} \mathrm{d}k \ \exp\big(-L^2k^2 + (2L^2k_0 - ix_0 + ix)k - L^2k_0^2\big) \\
&= \sqrt{\frac{\pi}{L^2}} \exp\big(L^2k_0^2\big) \exp\left(\frac{2L^2k_0 - ix_0 + ix)^2}{4L^2}\right) \\
&= \frac{\sqrt{\pi}}{L} \exp\big(L^2k_0^2\big) \exp\big(L^2k_0^2\big) \exp(ik_0(x-x_0)) \exp\left(\frac{-(x-x_0)^2}{4L^2}\right)
\end{aligned}$$

Clearly, this wavefunction peaks at $x = x_0$. The extra phase shift introduced simply slides it over by that factor.

Solution 3:

(a) Let us first write the phase in terms of the momentum and energy of the wave:

$$\frac{1}{\hbar}\phi = \hbar kx - \hbar\omega t = px - \frac{p^2}{2m}t$$

Where momentum undergoes the transformation

$$p' = p - mv$$

Thus, this must be equal to:

$$\begin{aligned}
\frac{1}{\hbar}\phi &= (p' + mv)(x' + vt) - \frac{(p' + mv)^2}{2m}t \\
&= p'x + \cancel{p'vt} + mvx' + mv^2t - \frac{p'^2}{2m}t - \cancel{p'vt} - \frac{1}{2}mv^2t \\
&= p'x + mv(x - vt) - \frac{p'^2}{2m}t + \frac{1}{2}mv^2t \\
&= p'x + mvx - mv^2t - E't + \frac{1}{2}mv^2t \\
&= p'x - E't + mvx - \frac{1}{2}mv^2t
\end{aligned}$$

The first two terms are the phase of the primed wavefunction. The last two terms gives rise to $f(x,t)$ to be:

$$f(x,t) = \exp\left(\frac{i}{\hbar}\left(mvx - \frac{1}{2}mv^2t\right)\right)$$

Solution 4: Starting off with definition of the probability density function and differentiating it (and using the

SE later):

$$\begin{aligned}
\frac{\partial\rho}{\partial x} &= \int_{-\infty}^{\infty}(\frac{\partial\Psi^*}{\partial x}\Psi + \Psi^*\frac{\partial\Psi}{\partial x})\,\mathrm{d}x \\
&= \frac{i}{\hbar}((\hat{H}\Psi)^*\Psi - \Psi^*(\hat{H}\Psi)) \\
&= \frac{\hbar}{2im}(\Psi\nabla^2\Psi^* - \Psi^*\nabla^2\Psi) \\
&= -\nabla\cdot(\frac{\hbar}{2im}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)) \\
&= -\nabla\cdot(\frac{\hbar}{m}\,\mathrm{Im}(\Psi^*\nabla\Psi))
\end{aligned}$$

Noting the charge conservation equation, it can be concluded that:

$$J(x,t) = \frac{\hbar}{m}\,\mathrm{Im}(\Psi^*(x,t)\nabla\Psi(x,t))$$

Solution 5: $|\gamma(t)|$ will continue increasing as the two wavepackets evolve, since their wavefunctions will have more overlap as they spread out.

Solution 6: First note that:

$$J(x) = \frac{\hbar}{m}\,\mathrm{Im}(\Psi^*(x)\frac{\partial\Psi(x)}{\partial x}) = \frac{\hbar}{2im}(\Psi^*(x)\frac{\partial\Psi(x)}{\partial x} - \Psi(x)\frac{\partial\Psi^*(x)}{\partial x})$$

Then,

(a) $\Psi(x) = Ae^{\gamma x}$:

$$\begin{aligned}
J(x) &= \frac{\hbar}{2im}(A^*e^{\gamma x}\frac{\partial}{\partial x}(Ae^{\gamma x}) - Ae^{\gamma x}\frac{\partial}{\partial x}(A^*e^{\gamma x})) \\
&= \frac{\hbar|A|^2}{2im}(e^{\gamma x}\frac{\partial}{\partial x}(e^{\gamma x}) - e^{\gamma x}\frac{\partial}{\partial x}(e^{\gamma x})) \\
&= 0
\end{aligned}$$

Alternatively,

$$\begin{aligned}
J(x) &= \frac{\hbar}{m}\,\mathrm{Im}(A^*e^{\gamma x}\frac{\partial}{\partial x}(Ae^{\gamma x})) \\
&= \frac{\hbar}{m}\,\mathrm{Im}(AA^*\gamma e^{2\gamma x}) \\
&= 0
\end{aligned}$$

(b) $\Psi(x) = N(x)e^{iS(x)/\hbar}$:

$$\begin{aligned}
J(x) &= \frac{\hbar}{2im}(N(x)e^{-iS(x)/\hbar}\frac{\partial}{\partial x}(N(x)e^{iS(x)/\hbar}) - N(x)e^{iS(x)/\hbar}\frac{\partial}{\partial x}(N(x)e^{-iS(x)/\hbar})) \\
&= \frac{1}{m}N(x)^2\frac{\partial S(x)}{\partial x}
\end{aligned}$$

Alternatively,

$$\begin{aligned}
J(x) &= \frac{\hbar}{m}\,\mathrm{Im}(N(x)e^{-iS(x)/\hbar}\frac{\partial}{\partial x}(N(x)e^{iS(x)/\hbar})) \\
&= \frac{\hbar}{m}\,\mathrm{Im}(N(x)^2 * i\frac{\partial S(x)}{\partial x}/\hbar) \\
&= \frac{1}{m}N(x)^2\frac{\partial S(x)}{\partial x}
\end{aligned}$$

(c) $\Psi(x) = Ae^{ikx} + Be^{-ikx}$:

$$\begin{aligned}
J(x) &= \frac{\hbar}{2im}((A^* e^{-ikx} + B^* e^{ikx})\frac{\partial}{\partial x}(Ae^{ikx} + Be^{-ikx}) - (Ae^{ikx} + Be^{-ikx})\frac{\partial}{\partial x}(A^* e^{-ikx} + B^* e^{ikx})) \\
&= \frac{\hbar k}{m}(AA^* - BB^*) \\
&= \frac{\hbar k}{m}(|A|^2 - |B|^2)
\end{aligned}$$

Alternatively,

$$\begin{aligned}
J(x) &= \frac{\hbar}{m}\,\mathrm{Im}((A^* e^{-ikx} + B^* e^{ikx})\frac{\partial}{\partial x}(Ae^{ikx} + Be^{-ikx})) \\
&= \frac{\hbar}{m}(AA^* - BB^*) \\
&= \frac{\hbar}{m}(|A|^2 - |B|^2)
\end{aligned}$$

Problem Set 4

Solution 1:

(a) For $\Delta x \approx 10^{-10}$ m, $t_s \approx \frac{m_p}{\hbar}(\Delta x)^2 \approx 2.5 \times 10^{-14}$ s . For $\Delta x \approx 0.01$ m, $t_s \approx \frac{m_p}{\hbar}(\Delta x)^2 \approx 252$ s

(b) For the wavepacket to be localized,

$$\frac{(\Delta p)^2 t}{m\hbar} \ll 1.$$

Thus, if we show that

$$\frac{(\Delta p)^2 t}{m\hbar} \approx \frac{\Delta p}{p}$$

it is sufficient to conclude that $\frac{\Delta p}{p} \ll 1$ means that the wavepacket is localized.

$$\begin{aligned}\frac{(\Delta p)^2 t}{m\hbar} &\approx \frac{(\Delta p)^2(\Delta x/(p/m))}{m\hbar} \\ &\approx \frac{(\Delta p \Delta x)\Delta p}{\hbar p} \\ &\approx \frac{\Delta p}{p}\end{aligned}$$

Solution 2:

(a) The probability current is defined as (in one-dimension):

$$\begin{aligned}J &= \frac{\hbar}{m}\operatorname{Im}\left(\Psi^* \frac{\partial}{\partial x}\Psi\right) \\ &= \frac{\hbar}{m}\operatorname{Im}\left(e^{-ikz}(ike^{ikz})\right) \\ &= \frac{\hbar}{m}\operatorname{Im}(ik) \\ &= \frac{\hbar k}{m}\end{aligned}$$

The flux through any closed surface will be zero as the probability current is constant.

(b) In spherical coordinates, the probability current is:

$$J = \frac{\hbar}{m}\operatorname{Im}\left(\Psi^* \nabla \Psi\right)$$

but since we are only concerned about the radial component, we have:

$$\begin{aligned}J &= \frac{\hbar}{m}\operatorname{Im}\left(\frac{f^*(\theta)f(\theta)}{r}e^{-ikr}\frac{\partial}{\partial r}\left(\frac{e^{ikr}}{r}\right)\right) \\ &= \frac{\hbar}{m}\operatorname{Im}\left(\frac{f^*(\theta)f(\theta)}{r}e^{-ikr}\left(-\frac{e^{ikr}}{r^2} + \frac{ike^{ikr}}{r}\right)\right) \\ &= \frac{\hbar}{m}\operatorname{Im}\left(\frac{f^*(\theta)f(\theta)ik}{r^2} - \frac{f^*(\theta)f(\theta)}{r}\right)\end{aligned}$$

The complex number is in the form of $bi + a$ so the imaginary part is just b or:

$$J = \frac{\hbar k}{m}\frac{f^*(\theta)f(\theta)}{r^2}$$

(c) Our wavefunction is now

$$\Psi = e^{ikz} + \frac{f(\theta)}{r}e^{ikr}$$

so the probability current is:

$$\begin{aligned}
J &= \frac{\hbar}{m}\,\mathrm{Im}\left(\left(e^{-ikz}+\frac{f^*(\theta)}{r}e^{-ikr}\right)\frac{\partial}{\partial r}\left(e^{ikz}+\frac{f(\theta)}{r}e^{ikr}\right)\right)\\
&= \frac{\hbar}{m}\,\mathrm{Im}\left(\left(e^{-ikz}+\frac{f^*(\theta)}{r}e^{-ikr}\right)\left(ik\cos\theta e^{ikz}+\frac{f(\theta)}{r}ike^{ikr}\right)\right)\\
&= \frac{\hbar}{m}\,\mathrm{Im}\left(\left(e^{-ikz}\right)\left(ik\cos\theta e^{ikz}\right)+\left(e^{-ikz}\right)\left(\frac{f(\theta)}{r}ike^{ikr}\right)+\left(\frac{f^*(\theta)}{r}e^{-ikr}\right)\left(ik\cos\theta e^{ikz}\right)\right)\\
&= \frac{\hbar}{m}\,\mathrm{Im}\left(ik\cos\theta+\frac{f(\theta)}{r}ike^{ikr(1-\cos\theta)}+\frac{f^*(\theta)}{r}ik\cos\theta e^{ikr(\cos\theta-1)}\right)\\
&= \frac{\hbar k}{mr}\,\mathrm{Im}\left(i\left(r\cos\theta+f(\theta)e^{ikr(1-\cos\theta)}+f^*(\theta)e^{ikr(\cos\theta-1)}\right)\right)
\end{aligned}$$

Solution 3:

(a) Use the normalization integral to confirm the wavefunction is normalized.

$$\int_{-\infty}^{\infty}|\Psi_a(x,0)|^2\,\mathrm{d}x=\int_{-\infty}^{\infty}\left(\frac{1}{(2\pi)^{1/4}\sqrt{a}}e^{-x^2/4a^2}\right)^2\,\mathrm{d}x=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}a}e^{-x^2/2a^2}\,\mathrm{d}x$$

Now, we can make the following u-substitution:

$$u=\frac{x}{a\sqrt{2}},\ \mathrm{d}u=\frac{a}{\sqrt{2}}\mathrm{d}x$$

Then,

$$\int_{-\infty}^{\infty}|\Psi_a(x,0)|^2\,\mathrm{d}x=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}e^{-u^2}\,\mathrm{d}u=1$$

This result satisfies the normalization condition (note: we used the standard result of the Gaussian integral for the calculation above)

(b) Using the inverse Fourier transform:

$$\begin{aligned}
\Phi_a(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi_a(x,0)\,e^{-ikx}\,\mathrm{d}x\\
&= \frac{1}{(2\pi)^{3/4}\sqrt{a}}\int_{-\infty}^{\infty}e^{-x^2/4a^2-ikx}\,\mathrm{d}x\\
&= \frac{1}{(2\pi)^{3/4}\sqrt{a}}\sqrt{4\pi a^2}e^{-k^2a^2}\\
&= \left(\frac{2a}{\pi}\right)^{1/4}e^{-a^2k^2}
\end{aligned}$$

(c) Since the particle is free, the potential is always zero. Thus,

$$E=\hbar\omega=\frac{(\hbar k)^2}{2m}\implies\omega=\frac{\hbar k^2}{2m}$$

Then, we can use the result from b) for $\Phi(k)$ to time evolve the wavepacket.

$$\begin{aligned}\Psi_a(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Phi_a(k)\, e^{i(kx-\omega t)}\, \mathrm{d}k \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}k\, \left(\frac{2a}{\pi}\right)^{1/4}\exp(-a^2k^2 + i(kx - \frac{\hbar k^2}{2m}t)) \\ &= \left(\frac{a}{2\pi^3}\right)^{1/4}\int_{-\infty}^{\infty}\mathrm{d}k\, \exp(-(a^2+\frac{i\hbar}{2m}t)k^2 + ixk) \\ &= \left(\frac{a}{2\pi^3}\right)^{1/4}\int_{-\infty}^{\infty}\mathrm{d}k\, \exp(-\frac{\hbar}{2m}(\tau+it)k^2 + ixk) \\ &= \left(\frac{a}{2\pi^3}\right)^{1/4}\sqrt{-\frac{2m}{\hbar(\tau+it)}}\exp\left(-\frac{mx^2}{2\hbar(\tau+it)}\right)\end{aligned}$$

Solution 4:

(a) Notice that

$$\int_{-\infty}^{\infty}\mathrm{d}x|\Psi(x)|^2 = \int_{-\infty}^{\infty}\mathrm{d}x\Psi(x)\Psi^*(x) = \int_{-\infty}^{\infty}\mathrm{d}x\left(\int_{-\infty}^{\infty}\mathrm{d}p\Phi(p)e^{ixp}\right)\left(\int_{-\infty}^{\infty}\mathrm{d}p'\Phi^*(p')e^{-ixp'}\right).$$

This can be rewritten with the dirac delta function, as $\int e^{ix(p-p')} = \delta(p-p')$, so

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}p\Phi(p)\int_{-\infty}^{\infty}\mathrm{d}p'\Phi^*(p')\delta(p-p') = \int_{-\infty}^{\infty}\mathrm{d}p|\Phi(p)|^2.$$

(b) Using the standard normalization integral and taking into account the spherical symmetry of the situation:

$$\int_0^{\infty}\left(Ne^{-r/a_0}\right)^2 4\pi r^2\,\mathrm{d}r = \int_0^{\infty}N^2e^{-2r/a_0}\,4\pi r^2\,\mathrm{d}r = \pi N^2 a_0^3 = 1$$

Thus,

$$N = \sqrt{\frac{1}{\pi a_0^3}}$$

Using Parseval's theorem gives us a normalization condition for momentum wavefunctions similar to the one used in part b):

$$\int_{-\infty}^{\infty}\frac{N'^2}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^4}\,4\pi p^2\,\mathrm{d}p = 1$$

Then,

$$\int_{-\infty}^{\infty}\frac{4\pi p^2\,\mathrm{d}p}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^4} = \frac{\pi^2\hbar^3}{8a_0^3} \implies N' = \sqrt{\frac{8a_0^3}{\pi^2\hbar^3}}$$

Using the probabalistic interpretation of the momentum wavefunction, the probability that the **magnitude** of the momentum exceeds $\hbar/a_0$ is:

$$\int_{\hbar/a_0}^{\infty}\frac{N'^2}{\left(1+\frac{a_0^2p^2}{\hbar^2}\right)^4}\,4\pi p^2\,\mathrm{d}p$$

Solution 5:

(a) First, we prove that $\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}\right\rangle$.

Proof. We can do this by first analyzing the average momentum

$$\langle p \rangle = -i\hbar \int \psi^* \frac{\partial \psi}{\partial x} dx.$$

Taking the derivative of this gets us

$$\begin{aligned} \frac{d}{dt}\langle p \rangle &= (-i\hbar)\frac{d}{dt}\int \psi^* \frac{\partial \psi}{\partial x} dx \\ &= (-i\hbar)\int \frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial x}\right) dx \\ &= (-i\hbar)\int \left[\frac{\partial \psi^*}{\partial t}\cdot\frac{\partial \psi}{\partial x} + \psi^* \frac{\partial}{\partial x}\frac{\partial \psi}{\partial t}\right] dx \end{aligned}$$

Note that the last step was performed via chain rule. Let us now analyze the Time Dependent Schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar\frac{\partial \psi}{\partial t}.$$

Carrying some manipulations on the Time Dependent Schrodinger equation gives us

$$\begin{aligned} \frac{\partial \psi}{dt} &= \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right) \\ &= \frac{-i}{\hbar}\left(\frac{-\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi\right) \\ &= \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi. \end{aligned}$$

We can now plug these results back into our first expression for $\frac{d}{dt}\langle p \rangle$.

$$\frac{d}{dt}\langle p \rangle = (-i\hbar)\int \left[\left(\frac{-i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{i}{\hbar}V\psi\right)\frac{\partial \psi}{\partial x} + \psi^* \frac{\partial}{\partial x}\left(\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi\right)\right] dx$$

Separating this integral gives us

$$\begin{aligned} \frac{d}{dt}\langle p \rangle &= (-i\hbar)\left(-\frac{i\hbar}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] - (i\hbar)\left(\frac{i}{\hbar}\right)\int \left[V\psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{d}{dx}(V\psi)\right] dx \\ &= \left(-\frac{\hbar^2}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] + \int \left[V\psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{d}{dx}(V\psi)\right] dx. \end{aligned}$$

Now, note that

$$\frac{d}{dx}(V\psi) = \frac{dV}{dx}\psi - V\frac{\partial \psi}{\partial x}.$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\langle p \rangle &= \left(-\frac{\hbar^2}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] + \int \left[V\psi^* \frac{\partial \psi}{\partial x} - \psi^* \frac{dV}{dx}\psi - V\psi^* \frac{\partial \psi}{\partial x}\right] dx \\ &= \left(-\frac{\hbar^2}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] + \int \left[\cancel{V\psi^* \frac{\partial \psi}{\partial x}} - \psi^* \frac{dV}{dx}\psi - \cancel{V\psi^* \frac{\partial \psi}{\partial x}}\right] dx \\ &= \left(-\frac{\hbar^2}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] + \int \left[-\psi^* \frac{dV}{dx}\psi\right] dx \end{aligned}$$

Now, this is where the exciting part happens. Note that

$$\int \left[-\psi^* \frac{dV}{dx}\psi\right] dx = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

therefore, rewriting the integral once again gives us

$$\frac{d}{dt}\langle p \rangle = \left(-\frac{\hbar^2}{2m}\right)\int \left[\frac{\partial^2 \psi^*}{\partial x^2}\frac{\partial \psi}{\partial x} - \psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial x^2}\right)\right] + \left\langle -\frac{\partial V}{\partial x}\right\rangle.$$

Let us analyze the second term of the integral. Via integration by parts, we find that

$$\begin{aligned}
\psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right) &= \psi^* \int \frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right) dx - \int \left[\frac{\partial\psi^*}{\partial x} \int \frac{\partial}{\cancel{\partial x}}\left(\frac{\partial^2\psi}{\partial x^2}\cancel{dx}\right)\right] dx \\
&= \psi^* \left.\frac{\partial^2\psi}{\partial x^2}\right|_{-\infty}^{+\infty} - \int \frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2} dx
\end{aligned}$$

The limit of the wavefunction tells us that when $x \to \pm\infty$, then $\psi \to 0$. This means that we are only left with

$$\psi^* \frac{\partial}{\partial x}\left(\frac{\partial^2\psi}{\partial x^2}\right) = -\int \frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2} dx.$$

Substituting this into our expression for $d\langle p\rangle/dt$ gives us a spectacular result.

$$\begin{aligned}
\frac{d}{dt}\langle p\rangle &= \left(-\frac{\hbar^2}{2m}\right) \int \left[\frac{\partial^2\psi^*}{\partial x^2}\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\frac{\partial^2\psi}{\partial x^2}\right] + \left\langle -\frac{\partial V}{\partial x}\right\rangle \\
&= \left(-\frac{\hbar^2}{2m}\right) \int \frac{\partial}{\cancel{\partial x}}\left(\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right)\cancel{dx} + \left\langle -\frac{\partial V}{\partial x}\right\rangle
\end{aligned}$$

Integrating this expression gives us

$$\frac{d}{dt}\langle p\rangle = \left(-\frac{\hbar^2}{2m}\right) \cdot \left.\frac{\partial\psi^*}{\partial x}\frac{\partial\psi}{\partial x}\right|_{-\infty}^{+\infty} + \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Once again, noting that the limit of the wavefunction tells us that when $x \to \pm\infty$, then $\psi \to 0$, we are left with

$$\boxed{\frac{d}{dt}\langle p\rangle = \left\langle -\frac{\partial V}{\partial x}\right\rangle}$$

□

Solution 6: We know that the expecation value of an operator is given by

$$\langle [\hat{p}, H] \rangle = i\hbar \frac{d\langle \hat{p} \rangle}{dt}$$

where we have chosen our operator to be our momentum operator $\hat{p} = -i\hbar \frac{\partial}{\partial x}$. It turns out that given $V = 0$, the momentum operator commutes with the Hamiltonian:

$$\begin{aligned}
[\hat{p}, H]f &= (\hat{p}H - H\hat{p})f \\
&= \hat{p}\left(\frac{\hat{p}^2}{2m} + V\right)f - \hat{p}\left(\frac{\hat{p}^2}{2m} + V\right)f \\
&= -i\hbar \frac{\partial f}{\partial x}(Vf) + Vi\hbar \frac{\partial f}{\partial x} \\
&= -i\hbar V \frac{\partial f}{\partial x} - i\hbar \frac{\partial V}{\partial x} f + i\hbar V \frac{\partial f}{\partial x} \\
&= -i\hbar \frac{\partial V}{\partial x}
\end{aligned}$$

If V is a constant, which is true for a free Schrodinger wave, then that implies $\langle \hat{p} \rangle$ is a constant and thus its eigenvalue is also a constant. We can similarly show that this result is true for $\langle \hat{p}^2 \rangle$ as well. Therefore:

$$\sigma_p = \langle p^2 \rangle - \langle p \rangle^2$$

is also a constant.

Solution 7:

(a) First, note that:

$$\begin{aligned}
x_0 = \langle x \rangle_{\psi_0} &= \int_{-\infty}^{\infty} \psi_0^*(x)\hat{x}\psi_0(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \psi_0^*(x)x\psi_0(x)\,\mathrm{d}x \\
p_0 = \langle p \rangle_{\psi_0} &= \int_{-\infty}^{\infty} \psi_0^*(x)\hat{p}\psi_0(x)\,\mathrm{d}x = -i\hbar \int_{-\infty}^{\infty} \psi_0^*(x)\frac{\partial}{\partial x}\psi_0(x)\,\mathrm{d}x
\end{aligned}$$

Now, in order to calculate the expectation values of the new wavefunction, first note that:

$$\psi_{new}(x) = \hat{B}_q\psi(x) = e^{iqx/\hbar}\ \psi_0(x)$$

(b)

$$\begin{aligned}
\langle x \rangle_{\psi_{new}} &= \int_{-\infty}^{\infty} \psi_{new}^*(x)\hat{x}\psi_{new}(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \psi_0^*(x)e^{-iqx/\hbar}x\psi_0(x)e^{iqx/\hbar}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \psi_0^*(x)x\psi_0(x)\,\mathrm{d}x \\
&= x_0
\end{aligned}$$

(c)

$$\begin{aligned}\langle p\rangle_{\psi_{new}} &= \int_{-\infty}^{\infty} \psi^*_{new}(x)\hat{p}\psi_{new}(x)\ \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \psi^*_0(x)e^{-iqx/\hbar}\hat{p}\psi_0(x)e^{iqx/\hbar}\ \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \psi^*_0(x)e^{-iqx/\hbar}(-i\hbar\frac{\partial}{\partial x})\psi_0(x)e^{iqx/\hbar}\ \mathrm{d}x \\ &= -i\hbar\int_{-\infty}^{\infty} \psi^*_0(x)e^{-iqx/\hbar}(e^{iqx/\hbar}\frac{\partial\psi_0(x)}{\partial x} + \frac{iq}{\hbar}e^{iqx/\hbar}\psi_0(x))\ \mathrm{d}x \\ &= -i\hbar\int_{-\infty}^{\infty} \psi^*_0(x)\frac{\partial}{\partial x}\psi_0(x)\ \mathrm{d}x + q\int_{-\infty}^{\infty} \psi^*_0(x)\psi_0(x)\ \mathrm{d}x \\ &= p_0 + q\end{aligned}$$

(d) The boost operator has no effect on the measurement of the position, since the magnitude of the wavefunction is not changing (that is the only relevant matter when calculating a position-space probability distribution). However, the expected value of the momentum is ”translated” by a constant value, based on the value q of the boost operator.

(e) For the momentum-boost commutator:

$$[\hat{p}, \hat{B}_q]\psi(x) = \hat{p}\hat{B}_q\psi(x) - \hat{B}_q\hat{p}\psi(x) = -i\hbar\frac{\partial}{\partial x}\left(e^{iqx/\hbar}\psi(x)\right) - e^{iqx/\hbar}(-i\hbar)\frac{\partial\psi(x)}{\partial x}$$

Using the product rule and expanding and cancelling terms:

$$[\hat{p}, \hat{B}_q]\psi(x) = q\psi(x) \implies [\hat{p}, \hat{B}_q] = q$$

For the position-boost commutator:

$$[\hat{x}, \hat{B}_q] = \hat{x}\hat{B}_q - \hat{B}_q\hat{x} = xe^{iqx/\hbar} - e^{iqx/\hbar}x = 0$$

Problem Set 5

Solution 1:

(a) First, note that, by the normalization condition, the following is true:

$$N^2 \int_{-\infty}^{\infty} \psi(x)\, \psi^*(x)\, \mathrm{d}x = N^2 \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^2}\right)\, \mathrm{d}x = 1$$

Also, note that by the symmetry of the situation, $\langle x \rangle$ must be zero. However, we can also show this through calculation.

$$\begin{aligned}
(\Delta x)^2 &= \langle x^2 \rangle - \langle x \rangle^2 \\
&= \int_{-\infty}^{\infty} \psi^*(x)\, \hat{x}^2\, \psi(x)\, \mathrm{d}x - \left(\int_{-\infty}^{\infty} \psi^*(x)\, \hat{x}\, \psi(x)\, \mathrm{d}x\right)^2 \\
&= N^2 \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{a^2}\right)\, \mathrm{d}x - \left(N^2 \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{a^2}\right) \mathrm{d}x\right)^2 \\
&= \frac{1}{2} a^2 N^2 \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^2}\right)\, \mathrm{d}x - 0 \\
&= \frac{1}{2} a^2
\end{aligned}$$

Similarly, it can be said that $\langle p \rangle$ is zero by the symmetry of the situation (prove this using the integral definition of expectation values if you wish, in a similar manner to how it was proved above).

$$\begin{aligned}
(\Delta p)^2 &= \langle p^2 \rangle - \langle p \rangle^2 \\
&= \int_{-\infty}^{\infty} \psi^*(x)\, \hat{p}^2\, \psi(x)\, \mathrm{d}x - 0 \\
&= -\hbar^2 N^2 \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2a^2}\right) \frac{\partial^2}{\partial x^2}\left(\exp\left(-\frac{x^2}{2a^2}\right)\right)\, \mathrm{d}x \\
&= -\frac{\hbar^2 N^2}{a^4} \int_{-\infty}^{\infty} (x^2 - a^2)\, \exp\left(-\frac{x^2}{a^2}\right) \mathrm{d}x \\
&= -\frac{\hbar^2 N^2}{2a^2} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^2}\right)\, \mathrm{d}x + \frac{\hbar^2 N^2}{a^2} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{a^2}\right) \mathrm{d}x \\
&= -\frac{\hbar^2}{2a^2} + \frac{\hbar^2}{a^2} \\
&= \frac{\hbar^2}{2a^2}
\end{aligned}$$

Then,

$$\Delta x \Delta p = \sqrt{\frac{a^2}{2} \frac{\hbar^2}{2a^2}} = \frac{\hbar}{2}$$

which does in fact saturate the Heisenberg uncertainty product.

(b)

$$\begin{aligned}
\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, \mathrm{d}x \\
&= \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2a^2} - \frac{ipx}{\hbar}\right) \mathrm{d}x \\
&= \frac{N}{\sqrt{2\pi\hbar}} \sqrt{2\pi a^2} \exp\left(-\frac{p^2 a^2}{2\hbar^2}\right) \\
&= \frac{Na}{\sqrt{\hbar}} \exp\left(-\frac{p^2 a^2}{2\hbar^2}\right)
\end{aligned}$$

We can verify this is correct by ensuring the normalization condition is met, as required by Parseval's Theorem.

$$\int_{-\infty}^{\infty} \phi(p)\phi^*(p)\, \mathrm{d}p = \frac{N^2a^2}{\hbar}\int_{-\infty}^{\infty} \exp\left(-\frac{p^2a^2}{\hbar^2}\right) \mathrm{d}p = N^2 a\sqrt{\pi} = 1$$

In fact, this is the same normalization condition obtained for the position wavefunction (which can be shown similarly), which lines up with Parseval's theorem and proves our wavefunction is correct. Then,

$$\begin{aligned}
(\Delta p)^2 &= \left\langle p^2 \right\rangle - \langle p \rangle^2 \\
&= \left\langle p^2 \right\rangle - 0 \\
&= \int_{-\infty}^{\infty} \phi^*(p)\, \hat{p}^2\, \phi(p)\, \mathrm{d}p \\
&= \frac{N^2a^2}{\hbar}\int_{-\infty}^{\infty} p^2 \exp\left(-\frac{p^2a^2}{\hbar^2}\right) \mathrm{d}p \\
&= \frac{\hbar^2}{2a^2}\frac{N^2a^2}{\hbar}\int_{-\infty}^{\infty} \exp\left(-\frac{p^2a^2}{\hbar^2}\right) \mathrm{d}p \\
&= \frac{\hbar^2}{2a^2}
\end{aligned}$$

This result lines up with what was calculated using the position wavefunction.

Solution 2: First, we prove the fact stated in the problem for $z = re^{i\theta}$:

$$\frac{\mathrm{Re}(z)}{|z|^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} = \frac{\cos(-\theta)}{r} = \mathrm{Re}\left(\frac{1}{z}\right)$$

Also, note by the normalization condition that:

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, \mathrm{d}x = N^2\int_{-\infty}^{\infty} \exp\left(\mathrm{Re}\left(-\frac{x^2}{2\Delta^2}\right)\right)^2 \mathrm{d}x = N^2\int_{-\infty}^{\infty} \exp\left(-\frac{x^2\,\mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x = 1$$

(a) For both momentum and position calculations, we can again note by symmetry that $\langle x \rangle = 0$ and $\langle p \rangle = 0$. Then,

$$\begin{aligned}
(\Delta x)^2 &= \left\langle x^2 \right\rangle - 0^2 \\
&= \int_{-\infty}^{\infty} \psi^*(x)\, \hat{x}^2\, \psi(x)\, \mathrm{d}x \\
&= \int_{-\infty}^{\infty} x^2\, |\psi(x)|^2\, \mathrm{d}x \\
&= N^2\int_{-\infty}^{\infty} x^2 \exp\left(\mathrm{Re}\left(-\frac{x^2}{2\Delta^2}\right)\right)^2 \mathrm{d}x \\
&= N^2\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2\,\mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x \\
&= \frac{|\Delta|^4}{2\,\mathrm{Re}\left(\Delta^2\right)} N^2\int_{-\infty}^{\infty} \exp\left(-\frac{x^2\,\mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x \\
&= \frac{|\Delta|^4}{2\,\mathrm{Re}\left(\Delta^2\right)}
\end{aligned}$$

$$
\begin{aligned}
(\Delta p)^2 &= \left\langle p^2 \right\rangle - 0^2 \\
&= \int_{-\infty}^{\infty} \psi^*(x)\, \hat{p}^2\, \psi(x)\, \mathrm{d}x \\
&= -\hbar^2 N^2 \int_{-\infty}^{\infty} \exp^*\left(-\frac{x^2}{2\Delta^2}\right) \frac{\partial^2}{\partial x^2}\left(\exp\left(-\frac{x^2}{2\Delta^2}\right)\right) \\
&= -\frac{\hbar^2 N^2}{\Delta^4} \int_{-\infty}^{\infty} \left(x^2 - \Delta^2\right) \exp\left(-\frac{x^2\, \mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x \\
&= -\frac{\hbar^2 N^2 |\Delta|^4}{2\Delta^4\, \mathrm{Re}\left(\Delta^2\right)} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2\, \mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x + \frac{\hbar^2 N^2}{\Delta^2} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2\, \mathrm{Re}\left(\Delta^2\right)}{|\Delta|^4}\right) \mathrm{d}x \\
&= -\frac{\hbar^2 |\Delta|^4}{2\Delta^4\, \mathrm{Re}\left(\Delta^2\right)} + \frac{\hbar^2}{\Delta^2}
\end{aligned}
$$

Letting $|\Delta|^4 = (\Delta^2)^*\Delta^2$ and $2\,\mathrm{Re}(\Delta^2) = (\Delta^2)^* + \Delta^2$, we can simplify this expression to

$$
(\Delta p)^2 = \frac{\hbar^2}{2\,\mathrm{Re}(\Delta^2)}
$$

(b)

$$
\begin{aligned}
\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ipx/\hbar}\, \mathrm{d}x \\
&= \frac{N}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2\Delta^2} - \frac{ipx}{\hbar}\right) \mathrm{d}x \\
&= \frac{N}{\sqrt{2\pi\hbar}} \sqrt{2\pi\Delta^2} \exp\left(-\frac{p^2\Delta^2}{2\hbar^2}\right) \\
&= \frac{N\Delta}{\sqrt{\hbar}} \exp\left(-\frac{p^2\Delta^2}{2\hbar^2}\right)
\end{aligned}
$$

This result can be shown to be correct using Parseval's Theorem:

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\Phi(p)|^2 dp \\
&= \frac{N^2|\Delta|^2}{\hbar} \int_{-\infty}^{\infty} \exp\left(-\frac{p^2\, \mathrm{Re}(\Delta^2)}{\hbar^2}\right) dp \\
&= N^2|\Delta|^2 \sqrt{\frac{\pi}{\mathrm{Re}(\Delta^2)}}
\end{aligned}
$$

This is the same normalization condition that we originally stated where we worked in position space. The uncertainty is thus

$$
\begin{aligned}
(\Delta p)^2 &= \int_{-\infty}^{\infty} p^2 |\Phi|^2 dp \\
&= \frac{N^2|\Delta|^2}{\hbar} \int_{-\infty}^{\infty} p^2 \exp\left(\frac{-p^2}{\hbar^2}\, \mathrm{Re}(\Delta^2)\right) dp \\
&= \frac{\hbar^2}{2\,\mathrm{Re}(\Delta^2)} \frac{N^2|\Delta|^2}{\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{-p^2}{\hbar^2}\, \mathrm{Re}(\Delta^2)\right) dp \\
&= \frac{\hbar^2}{2\,\mathrm{Re}(\Delta^2)}
\end{aligned}
$$

where we have used our earlier normalization condition.

(c) We have:

$$
\begin{aligned}
(\Delta x \Delta p)^2 &= \frac{\hbar^2}{4}\left(\frac{|\Delta|^4}{\mathrm{Re}(\Delta^2)^2}\right) \\
&= \frac{\hbar^2}{4}\left(\frac{|\Delta|^4}{\mathrm{Re}(|\Delta|^2 e^{i2\Phi_\Delta})^2}\right) \\
&= \frac{\hbar^2}{4}\left(\frac{|\Delta|^4}{(|\Delta|^2\cos(2\phi_\Delta))^2}\right) \\
\Delta x \Delta p &= \frac{\hbar}{2\cos(2\Phi_\Delta)}
\end{aligned}
$$

For $\Phi_\Delta = 0$, then Δ becomes real and it becomes identical to the first problem. For $\Phi_\Delta \to \pi/4$, the product $\Delta x \Delta p \to \infty$. This is because at this limit, Δ^2 becomes purely imaginary. Thus, we get a pure phase with a constant oscillation and no dampening effect, causing there to be infinite momentum and position eigenstates with the same probability, causing their uncertainty to become infinite as well.

(d) In problem three, the given wavefunction was:

$$\Psi = N\exp\left(-\frac{x^2}{4a^2}\right)$$

Therefore, $\Delta^2 = 2a^2$. The uncertainty at $t = 0$ is then:

$$(\Delta p)^2 = \frac{\hbar^2}{2a^2}$$

Including the time dependence, the wavefunction becomeS:

$$\Psi(x,t) = N\exp\left(-\frac{mx^2}{2\hbar(\tau + it)}\right)$$

where $\tau = \frac{2ma^2}{\hbar}$. Therefore:

$$\Delta^2 = \frac{\hbar(\tau + it)}{m}$$

and

$$\mathrm{Re}(\Delta^2) = \frac{\hbar\tau}{m} \implies \Delta^2 = 2a^2$$

which is equivalent to the time $t = 0$ case, showing that the momentum uncertainty is indeed time independent.

Solution 3:

(a) Substituting the given wavefunction into the time-dependent Schrodinger equation:

$$
\begin{aligned}
i\hbar\frac{\partial\Psi_n(x,t)}{\partial t} &= -\frac{\hbar^2}{2m}\frac{\partial^2\Psi_n(x,t)}{\partial x^2} \\
i\hbar\frac{\partial}{\partial t}\left(\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n(t)}\right) &= -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\left(\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n(t)}\right) \\
i\hbar\left(\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n(t)}\right)(-i\phi_n(t)\phi)\frac{d\phi_n(t)}{dt} &= \frac{\hbar^2}{2m}\left(\sin\left(\frac{n\pi}{a}x\right)e^{-i\phi_n(t)}\right)\left(\frac{n\pi}{a}\right)^2 \\
\int \phi_n d\phi &= \int \frac{h\pi^2 n^2}{2ma^2}dt \\
\phi_n &= \frac{\pi n}{a}\sqrt{\frac{ht}{m}}
\end{aligned}
$$

where we have assumed $\phi_n(0) = 0$.
To solve for the normalization constant, we can ignore the phase (it does not affect the magnitude of the

wavefunction). Setting the normalization integral to unity:

$$\int_0^a N^2 \sin^2\left(\frac{n\pi}{a}x\right) \mathrm{d}x = \frac{aN^2}{2} = 1 \implies N = \sqrt{\frac{2}{a}}$$

(b) The expectation value of $\langle x \rangle$ is:

$$\begin{aligned}\langle x \rangle &= N^2 \int_0^a x \sin^2\left(\frac{n\pi}{a}x\right) \\ &= \frac{a}{2}\end{aligned}$$

as expected, since there is even symmetry around the center. To determine $\langle x^2 \rangle$, we perform the same operation to get

$$\langle x^2 \rangle = \frac{a^2\left(2\pi^2 n^2 - 3\right)}{6\pi^2 n^2}$$

Therefore:

$$(\Delta x)^2 = \frac{a^2\left(2\pi^2 n^2 - 3\right)}{6\pi^2 n^2} - \frac{a^2}{4}$$

(c) To calculate Δp, we do it in a similar way. The momentum operator is:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

so we have:

$$\langle p \rangle = 0$$

and

$$\langle p^2 \rangle = (\Delta p)^2 = \frac{\hbar^2 n^2 \pi^2}{a^2}$$

(d) The uncertainty principle gives us:

$$\Delta x \Delta p = \frac{\hbar}{2}\sqrt{\frac{\pi^2 n^2 - 6}{3}}$$

This is minimized when $n = 1$, which satisfies the uncertainty principle.

(e) Since ϕ is real and $V(x)$ is a constant, then both the momentum uncertainty and position uncertainty will not change with respect to time.

Solution 4: First, note that due to the infinite potential:

$$\psi(x) = 0,\ x \leq 0$$

When the potential is zero, the time-independent Schrodinger equation can be rearranged and solved like so:

$$\frac{\partial^2\psi(x)}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi(x) \implies \psi(x) = \psi(x) = A\cos kx + B\sin kx,\ k^2 = \frac{2mE}{\hbar^2}$$

Note that the cosine term has to be zero in order for the wavefunction to satisfy the continuity condition $\psi(x) = 0$, since cosine is nonzero at $x = 0$. Thus, the final result is:

$$\psi(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ A\sin kx, & \text{for } x > 0 \end{cases}$$

where A is a constant and $k^2 = \frac{2mE}{\hbar^2}$

Solution 5: Remember that $E \geq 0$ because $V(x)$ has a minimum value of zero. With that said, for $x \leq 0$, $V(x) = 0$, so:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi(x) \implies \psi(x) = A\cos kx + B\sin kx,\ k^2 = \frac{2mE}{\hbar^2}$$

Note that the complex exponential can also be represented as a sum of sines and cosines. We will come back to this later when considering boundary conditions

For $x \geq 0$, $V(x) = V_0$, so:

$$\frac{\partial^2 \psi(x)}{\partial x^2} = -\frac{2m}{\hbar^2}\left(E - V_0\right)\psi(x)$$

Since $E - V_0 < 0$, it is clear that

$$\psi(x) = Ce^{-\kappa x},\ \kappa^2 = -\frac{2m}{\hbar^2}\left(E_0 - V\right)$$

Notice for the second case, we only care about the negative exponential term, since it must decay as x approaches ∞. Equating boundary conditions gives $A = C$. We also need to ensure that the derivative of the wavefunction is zero, which implies that $Bk = -C\kappa$, implying that $B = -C\frac{\kappa}{k} = -A\frac{\kappa}{k}$. Making these substitutions,

$$\psi(x) = \begin{cases} A\cos kx - A\frac{\kappa}{k}\sin kx, & \text{for } x \leq 0 \\ Ae^{-\kappa x}, & \text{for } x > 0 \end{cases}$$

where $k^2 = \frac{2mE}{\hbar^2}$ and $\kappa^2 = \frac{2m}{\hbar^2}\left(E_0 - V\right)$.

Solution 6: We break into two cases: $0 \leq x < a$ and $a \leq x$. For the first case, the Schrodinger equation gives

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{2m}{\hbar^2}(V - E)\Psi$$

Letting $k^2 = \frac{2m}{\hbar^2}(V - E)$, we get:

$$\Psi = \sin(kx)$$

For $x \geq a$, our Schrodinger equation gives

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{2mE}{\hbar^2}\Psi$$

instead. Letting $\kappa^2 = \frac{2mE}{\hbar^2}$, we get:

$$\Psi = Ae^{-kx}$$

Letting $\eta = ak$ and $\xi = a\kappa$, then we get:

$$z_0^2 = \eta^2 + \xi^2$$

which is the equation of a circle of the coordinate axes are η and ξ. At the boundary, Ψ must be continuous so

$$\sin(ka) = Ae^{-\kappa a}$$

Taking the derivative, we get:

$$k\cos(ka) = -\kappa Ae^{-\kappa a}$$

Dividing through, we get:

$$\xi = -\eta\cot\eta$$

To have three solutions, we need the lines $\xi = -\eta\cot\eta$ and $z_0^2 = \eta^2 + \xi^2$ to intersect three times for positive values. We see that this must mean that the minimum value of z_0 is

$$z_0 = \frac{5\pi}{2}$$

This is analogous to the odd case of the finite square well where both sides have a finite energy.d

Solution 7: As before, let us let $\xi \equiv a\kappa = \sqrt{\frac{2ma_0E}{\hbar^2}} = 1$. Due to the symmetry of the hydrogen atom, we wish to find even solutions, giving:

$$\xi = \eta \tan \eta$$

and

$$z_0^2 = \eta^2 + \xi^2$$

We have two unknowns z_0 and η so solving for z_0 gives

$$z_0 = 1.3192$$

It's easy to show that

$$z_0^2 = \frac{V_0}{E}$$

so

$$V_0 = z_0^2 E = 23.67 \text{ eV}$$

Problem Set 6

Solution 1:

(a) Note that the Hamiltonian energy operator $\hat{H}$ is proportional to the second position derivative of the wavefunction. Differentiating Ψ twice:

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{4\pi^2}{a^2}\sqrt{\frac{2}{3a}}\sin\frac{2\pi x}{a} - \frac{18\pi^2}{a^2}\sqrt{\frac{1}{3a}}\sin\frac{3\pi x}{a}$$

Note that this is **not** proportional to the initial wavefunction, meaning that it is **not** an energy eigenstate. Since the Schrodinger equation is linear, we can time evolve the wavefunction as:

$$\Psi(x,t) = \frac{1}{\sqrt{3}}\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)e^{-iE_1 t} + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{a}}\sin\left(\frac{3\pi x}{a}\right)e^{-iE_2 t}$$

(b) Note that although the wavefunction itself is not an energy eigenstate, it is composed of two sine functions, both of which are in fact energy eigenstates. Noting that $\hat{H}\Psi = E\Psi$, the energies of each of these eigenstates would be

$$E_{\Psi_1} = \frac{4\hbar^2\pi^2}{2ma^2}$$

and

$$E_{\Psi_2} = \frac{9\hbar^2\pi^2}{2ma^2}$$

Since we have the superposition of two energy eigenstates, we get:

$$\Psi = a_1\Psi_1 + a_2\Psi_2$$

and since the wavefunction is normalized, the probabilities to measure the two above energies are $\frac{1}{3}$ and $\frac{2}{3}$ respectively.

(c) $\int_{-\infty}^{\infty}$

Solution 2: Following the hint, the solution is fairly straightforward. Assume that Ψ_1 and Ψ_2 have the same energy, then:

$$\Psi_2\hat{H}\Psi_1 = \Psi_2 E\Psi_1$$
$$\Psi_1\hat{H}\Psi_2 = \Psi_1 E\Psi_2$$

Subtracting the two:

$$\Psi_2\frac{\partial^2\Psi_1}{\partial x^2} - \Psi_1\frac{\partial^2\Psi_2}{\partial x^2} = 0$$

This looks like the result of a product rule, so we can rewrite it as:

$$\frac{\partial}{\partial x}\left(\Psi_2\frac{\partial\Psi_1}{\partial x} - \Psi_1\frac{\partial\Psi_2}{\partial x}\right) = 0 \implies \Psi_2\frac{\partial\Psi_1}{\partial x} - \Psi_1\frac{\partial\Psi_2}{\partial x} = C$$

We know that both Ψ_1 and Ψ_2 vanish at infinity, so we must have $C = 0$. We then get:

$$\Psi_2 d\Psi_1 = \Psi_1 d\Psi_2$$

and solving this differential equation, we get:

$$\Psi_2 = c\Psi_1$$

Since Ψ_2 is a multiple of Ψ_1, the two solutions are not distinct.

Solution 3:

(a) The differential equation we get is:

$$(E_x + E_y)\Psi = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi$$

We can break this up into the x and y solutions, and then multiply them together to get:

$$\Psi = \frac{2}{\sqrt{L_x L_y}} \sin\left(\frac{n_x \pi x}{L_x}\right) \sin\left(\frac{n_y \pi y}{L_y}\right)$$

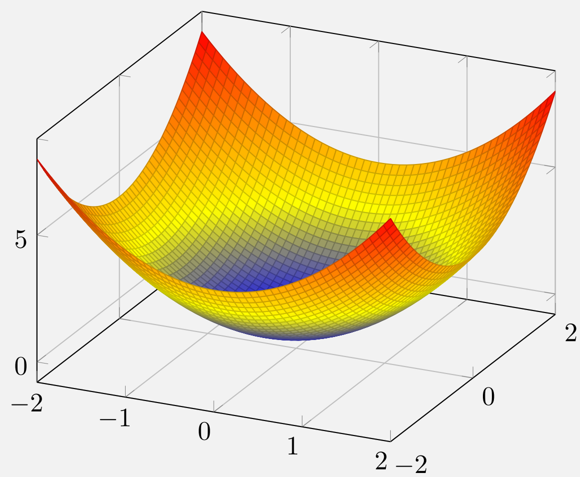
such that the energy eigenstates are

$$E = \frac{\hbar^2 \pi^2 n_x^2}{2mL_x^2} + \frac{\hbar^2 \pi^2 n_y^2}{2mL_y^2}$$

(b) If $L_x = L_y = L$, we get:

$$E = \frac{\hbar^2 \pi^2}{2mL^2}(n_x^2 + n_y^2)$$

and if n_x and n_y are plotted, they trace out a parabaloid. Given a particular circular cross section, all points on the circumference with integer coordinates represent degenerate energies.



This is now essentially a number theory problem involving diophantine equations. While a pattern can emerge by repeatedly applying the Brahmagupta–Fibonacci identity, these do not really hold any physical insight. For example:

$$(n_x, n_y) = (33, 56)$$

gives the same energy as

$$(n_x, n_y) = (63, 16).$$

However, there are also degenerate energies that are not coincidences. For example, if the ordered pair (n_x, n_y) gives a certain energy, then (n_y, n_x) also gives the same energy. This can be interpreted as a rotation of the wavefunction, meaning that if the x and y components are swapped, it would make no difference.

(c) This in a sense, is mainly a number theory problem. We want to show that:

$$\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} = \frac{n_x'^2}{L_x^2} + \frac{n_y'^2}{L_y^2}$$

has no nontrivial integers solutions given that $(L_x/L_y)^2$ is irrational. We will do this via contradiction. Suppose $k \equiv (L_x/L_y)^2$ is irrational and there are integer solutions (which represent degeneracies), then:

$$n_x^2 + kn_y^2 = n_x'^2 + kn_y'^2 \implies k = \frac{n_x^2 - n_x'^2}{n_y'^2 - n_y^2}$$

Since n is an integer, then the numerator and denominators must also be integers, and thus k is rational, contradicting our claim.

Solution 4:

(a) For $0 < x < a$, we have:

$$\frac{\partial^2}{\partial x^2}\Psi = -\frac{2mE}{\hbar^2} \equiv= -k^2\Psi$$

and the solution is:

$$\Psi_1 = \sin(kx)$$

For $a < x < 2a$, we have:

$$\frac{\partial^2}{\partial x^2}\Psi = -\frac{2m}{\hbar^2}(E - V)\Psi \equiv -\kappa^2\Psi$$

where $\kappa^2 < 0$. This solution is in the exponential form of:

$$\Psi_2 = Ae^{-kx} + Be^{kx}$$

We have three boundary conditions:

- $\Psi_1(a) = \Psi_2(a)$
- $\Psi_1'(a) = \Psi_2'(a)$
- $\Psi_2'(2a) = 0$

Using these boundary conditions, we get the equations:

$$\sin(ka) = Ae^{-\kappa a} + Be^{\kappa a}$$
$$k\cos(ka) = -\kappa Ae^{-\kappa a} + \kappa Be^{\kappa a}$$
$$0 = Ae^{-2\kappa a} + Be^{2\kappa a}$$

Rearranging the last equation, we get:

$$A = -Be^{4\kappa a}$$

and substituting this into the first two equations gives:

$$\sin(ka) = -Be^{3\kappa a} + Be^{\kappa a}$$
$$k\cos(ka) = \kappa Be^{3\kappa a} + \kappa Be^{\kappa a}$$

Dividing these two equations then give us:

$$\frac{1}{k}\tan(ka) = \frac{1}{\kappa}\frac{e^{\kappa a} - e^{3\kappa a}}{e^{\kappa a} + e^{3\kappa a}}$$

and makng everything a dimensionless number, we are left with:

$$\tan(\eta) = \frac{\eta}{\xi}\frac{e^{\xi} - e^{3\xi}}{e^{\xi} + e^{3\xi}}$$

As with usual, we also have:

$$z_0^2 = \eta^2 + \xi^2$$

and the number of possible energy levels correspond to the number of solutions between these two equations.

(b) Letting $z_0 = 2\pi$ gives 2 solutions. We can determine their energies as:

$$E = \left(\frac{\eta}{2\pi}\right)^2$$

giving $E = 0.18436V_0$ and $E = 0.70747V_0$ respectively.

Solution 5:

(a) Our differential equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + (V(x) - E)\Psi$$

Defining $x = au$, then we have $d(au)^2 = a^2(du^2)$. Then dividing through V_0 we define the characteristic energy as $e \equiv \frac{E}{V_0}$, getting:

$$-\frac{\hbar^2}{2mV_0a^2}\frac{d^2\Psi}{du^2} + (v(x) - e)\Psi$$

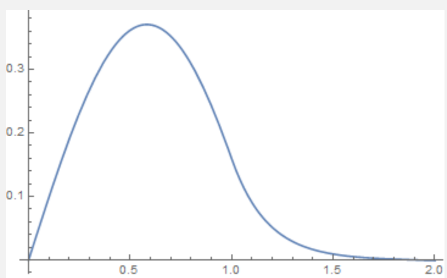
where:

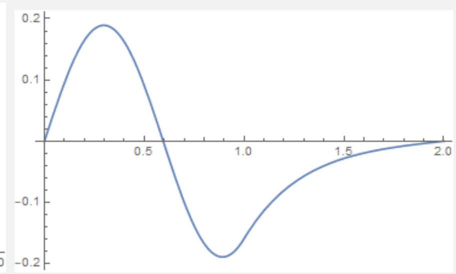
$$v(x) = \begin{cases} 0 & 0 < u < 1 \\ 1 & 1 < u < 2 \end{cases}$$

Notice that coefficient for the first term is just $\frac{1}{z_0^2}$. Therefore, our final dimensionless differential equation is:

$$-\frac{1}{z_0^2}\Psi'' + (v(x) - e)\Psi = 0$$

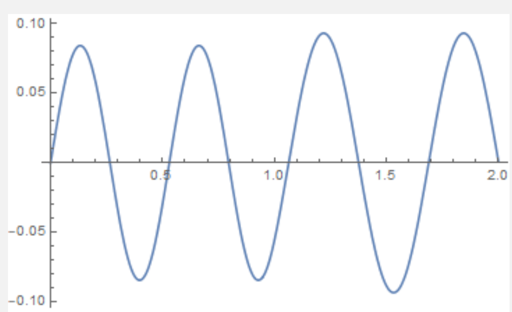
We can solve this via mathematica to obtain E_1 and E_2.





The next two energies (for when $E > V_0$) occurs approximately at $e = 1.205$ and $e = 1.563$.

(b) The eighth excited state has an energy of $e = 3.558$, corresponding to the following wavefunction:



We have approximately $A_L = 0.85$ and $A_R = 0.9$ such that $\frac{A_L}{A_R} \approx 0.94$ which is extremely close to the De Broglie approximation, where the ratio would be equal to one.

Solution 6:

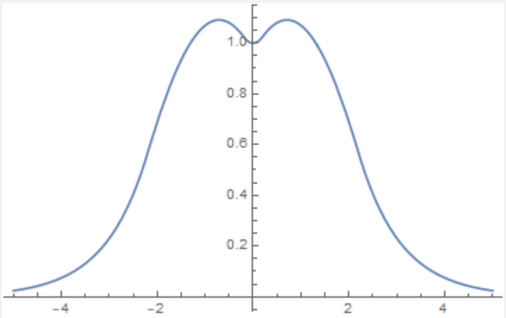
(a) The differential equation is once again

$$-\frac{1}{z_0^2}\Psi'' + (v(x) - e)\Psi = 0$$

with:

$$v(x) = \begin{cases} 0 & = x < -(2+\gamma) \\ -1 & = -(2+\gamma) < x < -\gamma \\ 0 & = -\gamma < x < \gamma \\ -1 & = \gamma < x < 2+\gamma \\ 0 & = x > \gamma + 2 \end{cases}$$

Solving this numerically via the Shooting method gives $e = -0.72936$ or $E = -17.26$ eV. The wavefunction looks like:



(b) The energy due to repulsion is $|E_0| = 13.6$ eV so the binding energy is:

$$E_b = |E_0| + E = -3.66 \text{ eV}$$

According to Griffiths, the experimentally determined binding energy is -2.8 eV so this model is rather accurate.

Problem Set 7

Solution 1:

(a) We have three regions to consider: $x < -a$, $-a < x < a$, and $x > a$. For the outer two regions, we have $V = 0$, and thus it would be experiencing exponential decay of:

$$\Psi = \pm e^{-\kappa|x|}$$

where $\kappa^2 \equiv -\frac{2mE}{\hbar^2}$ and the $\pm$ is dependent on whether the bound state is even or odd. For the middle region, the wavefunction can be either even or odd. If it's even, then it is given by:

$$\Psi = A_1 \cosh(\kappa x)$$

and if it's odd:

$$\Psi = A_2 \sinh(\kappa x)$$

Starting with the ground state (even), we can integrate the function from $-a-\epsilon$ to $-a+\epsilon$ with respect to x, and taking the limit as $\epsilon \to 0$:

$$-\frac{\hbar^2}{2m}\int_{-a-\epsilon}^{-a+\epsilon} \frac{d^2\Psi(x)}{dx^2}\,dx + \int_{-a-\epsilon}^{-a+\epsilon} (-g\delta(x+a))\Psi(x)\,dx = \int_{-a-\epsilon}^{-a+\epsilon} E\Psi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left(\frac{d\Psi}{dx}\bigg|_{-a+\epsilon} - \frac{d\Psi}{dx}\bigg|_{-a-\epsilon}\right) - g\delta(-a) = 0$$

$$-\frac{\hbar^2}{2m}\Delta_{-a}\Psi' - g\Psi(-a) = 0$$

$$\Delta_{-a}\Psi' = -\frac{2mg\Psi(-a)}{\hbar^2}$$

We attempt to evaluate the discontinuity function at $x = -a$. Due to symmetry, this will be equal to the discontinuity function at $x = a$. The function however, is continuous at $x = -a$, allowing us to determine A_1:

$$\Psi(-a) = A_1\cosh(-\kappa a) = -e^{-\kappa a} \implies A_1 = -\frac{e^{-\kappa a}}{\cosh(-\kappa a)}$$

allowing us to evaluate the discontinuity and solve for κ

$$\Delta_{-a}\Psi' = A\kappa\sinh(-\kappa a) - \left(-\kappa e^{-\kappa a}\right)$$

$$-\frac{2mg}{\hbar^2}\left(-e^{-\kappa a}\right) = \left(-e^{-\kappa a}\right)\kappa\tanh(-\kappa a) - \kappa\left(-e^{-\kappa a}\right)$$

$$-\frac{2mg}{\hbar^2} = \kappa\tanh(-\kappa a) - \kappa$$

$$\frac{2mg}{\hbar^2} = \kappa(1+\tanh(\kappa a)$$

$$2\lambda = a\kappa(1+\tanh(a\kappa))$$

We can define

$$a\kappa = \sqrt{\frac{2m|E|a^2}{\hbar^2}} = \sqrt{2e}$$

where e is the characteristic energy. This reduces our equation to:

$$2\lambda = \sqrt{2e}(1+\tanh\left(\sqrt{2e}\right))$$

Next, we look at our odd states. Everything is the same, except the discontinuity equation is different: κ

$$\Delta_{-a}\Psi' = A\kappa\cosh(-\kappa a) - \left(-\kappa e^{-\kappa a}\right)$$

$$-\frac{2mg}{\hbar^2}\left(-e^{-\kappa x}\right) = \left(-e^{-\kappa x}\right)\kappa\coth(-\kappa a) - \kappa\left(-e^{-\kappa x}\right)$$

$$-\frac{2mg}{\hbar^2} = \kappa\coth(-\kappa a) - \kappa$$

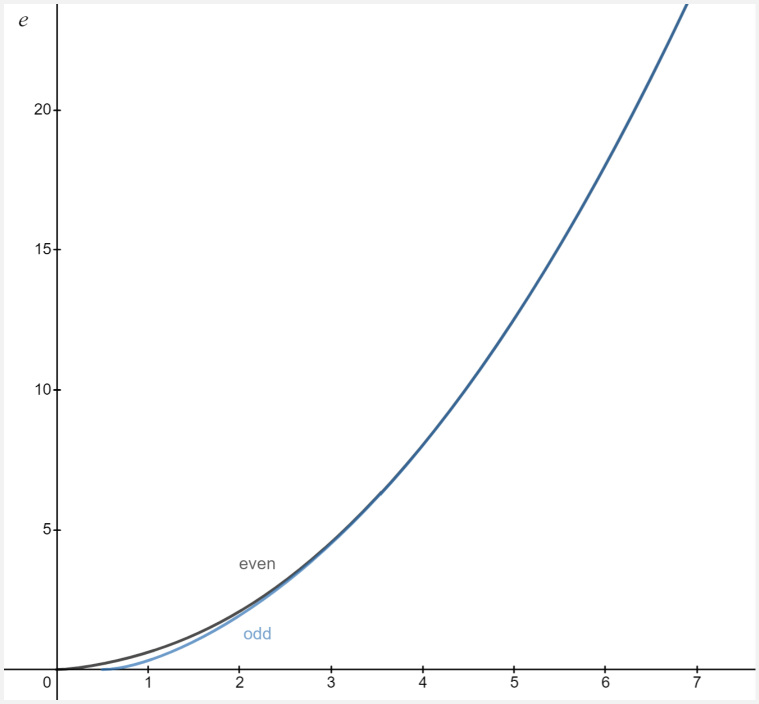
$$\frac{2mg}{\hbar^2} = \kappa(1+\coth(\kappa a)$$

$$2\lambda = a\kappa(1+\coth(a\kappa))$$

And again, our dimensionless equation which we can plot for e is:

$$2\lambda = \sqrt{2e}(1 + \coth\left(\sqrt{2e}\right))$$

When these two are graphed, we see the energies are both concave up with respect to the dimensionless parameter λ:



(b) As $2a \to \infty$, $a\kappa = \sqrt{2e}$ will approach infinity as well. Since $\tanh(x)$ and $\coth(x)$ have the same asymptotic value, we are left with:

$$\frac{2mga}{\hbar^2} = a\sqrt{\frac{2m|E|}{\hbar^2}} \implies E = -\frac{2mg^2}{\hbar^2}$$

which is the energy of a single delta function potential well. We can intuitively explain this in the framework of locality: The two potential wells are too far away from each other to "talk" or communicate with each other so in their own respective surroundings, the wavefunction behaves exactly as if they were the only delta functions.

However, a more interesting calculation would be the difference in energy as it approaches $2a$ grows large, but not necessarily at infinity. We have:

$$2\lambda = \sqrt{2e_1}(1 + \tanh\left(\sqrt{2e_1}\right))$$
$$2\lambda = \sqrt{2e_2}(1 + \coth\left(\sqrt{2e_2}\right))$$

Subtracting, we can write $e_1 = e_2 + \Delta e$ where $\Delta e \ll e_1$. Additionally, we can also expand $\tanh(x) \approx 1 - e^{-2x}$ and $\coth(x) \approx 1 + 2e^{-2x}$ for large x. This gives:

$$0 = \sqrt{2(e_2 + \Delta e)}(1 + \tanh\left(\sqrt{2e_1}\right)) - \sqrt{2e_2}(1 + \coth\left(\sqrt{2e_1}\right))$$
$$0 = \sqrt{2e_2}(1 + \tanh\left(\sqrt{2e_1}\right)) + \frac{\Delta e}{\sqrt{2e_2}}(1 + \tanh\left(\sqrt{2e_1}\right)) - \sqrt{2e_2}(1 + \coth\left(\sqrt{2e_1}\right))$$
$$0 = \sqrt{2e_2}\left(\tanh\left(\sqrt{2e_1}\right) - \coth\left(\sqrt{2e_2}\right)\right) + \frac{2\Delta e}{\sqrt{2e_2}}$$
$$0 = -\sqrt{2e_2}\left(e^{-\sqrt{2e_2}}e^{-\frac{\Delta e}{\sqrt{2e_2}}} + 2e^{-\sqrt{2e_2}}\right) + \frac{2\Delta e}{\sqrt{2e_2}}$$
$$0 = -3\sqrt{2e_2}e^{-\sqrt{2e_2}} + \frac{2\Delta e}{\sqrt{2e_2}}$$
$$\Delta e = 3e_2 e^{-\sqrt{2e_2}}$$

We can let $\lambda \approx \sqrt{2e_2}$ to get:

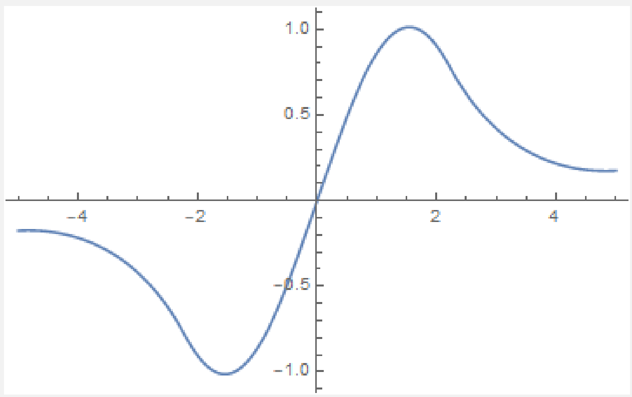
$$\Delta e = \frac{3}{2}\lambda^2 e^{-\lambda}$$

We can verify that as $\lambda \to \infty$, we indeed find that $\Delta e \to 0$.

Solution 2:

(a) For $b = 0$, we have the finite square well where the wavefunction is sinusoidal inside and exponentially decaying outside.

For $b = a$, we have the double square well problem similar to question 6 on problem set 6, where we have drawn what the first even solution looks like. The first odd solution will look like:



For $b \gg a$, we have a similar situation except the two wells will be not affected (or very little) by each other. Outside the wells, the wavefunction quickly goes to near zero.

(b) As discussed, it makes sense for the energies of both even and odd states to approach the same value V_0 as the separation b goes to infinity. The first even state has a negative energy so its energy starts below V_0. The first odd state has a positive energy so its total energy starts above V_0. Thus E_1 is monotonically increasing while E_2 is monotonically decreasing.

(c) Since systems tend to seek a lower energy level, so an electron in the even state tends to draw the protons together and an electron in the odd state does the opposite.

Problem Set 8

Solution 4:

(a) For a wavepacket built with energies $E < V_0$, we remember that $B/A = -e^{2i\delta(E)}$ where B and A represent the reflected and transmitted components respectively. This then means that the associated reflected wavepacket is given as

$$\Psi_{\text{ref}} = -\sqrt{a}\int_0^{\hat{k}} dk \Phi(k) e^{2i\delta(E)} e^{-ikx} e^{-iE(k)t/\hbar}.$$

(b)